Stat 20: Discussion at section #2

B. M. Bolstad, bolstad@stat.berkeley.edu

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Question 1

Roll two six sided dice. Let event A be that the sum of the digits on the dice is 9. Let event B be that doubles (same digits on both dice) were rolled. What is $P(A)$,$P(B)$,$P(A \cap B)$, $P(A \cup B)$, $P(A^c)$, $P(B^c)$, $P(A^c \cap B^c)$ and $P(A^c \cup B^c)$?

Answer: First let's list the possible outcomes such that (x_1, x_2) represents the outcome x_1 from the first of the dice and x_2 represents the outcome on the second. Then the possible outcomes are

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

The possible outcomes that are part of A are (3,6), (4,5), (5,4) and (6,3). There are 36 different possible outcomes. Therefore $P(A) = \frac{4}{36} = \frac{1}{9}$

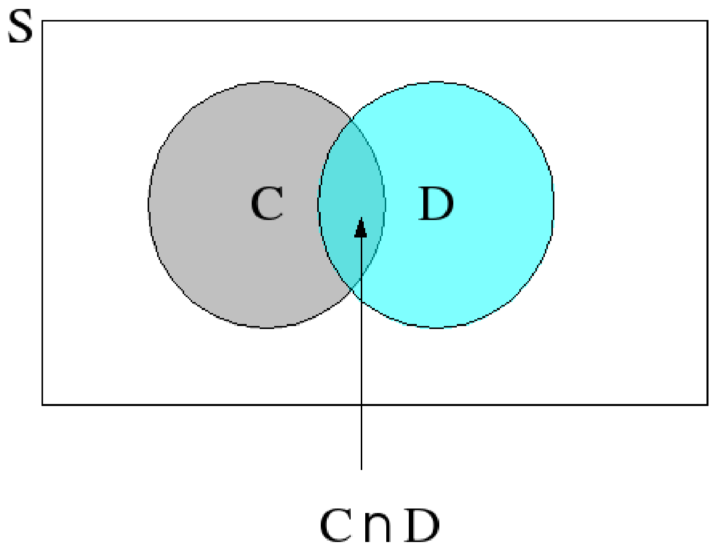
The possible outcomes that are part of B are (1,1), (2,2), (3,3), (4,4), (5,5) and (6,6). Therefore $P(B) = \frac{6}{36} = \frac{1}{6}$.

Now returning to our table of all possible outcomes, we highlight outcomes in event A by *italics* and outcomes in event B by **bold**.

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
(1,2)	**(2,2)**	(3,2)	(4,2)	(5,2)	(6,2)
(1,3)	(2,3)	**(3,3)**	(4,3)	(5,3)	*(6,3)*
(1,4)	(2,4)	(3,4)	**(4,4)**	*(5,4)*	(6,4)
(1,5)	(2,5)	(3,5)	*(4,5)*	**(5,5)**	(6,5)
(1,6)	(2,6)	*(3,6)*	(4,6)	(5,6)	**(6,6)**

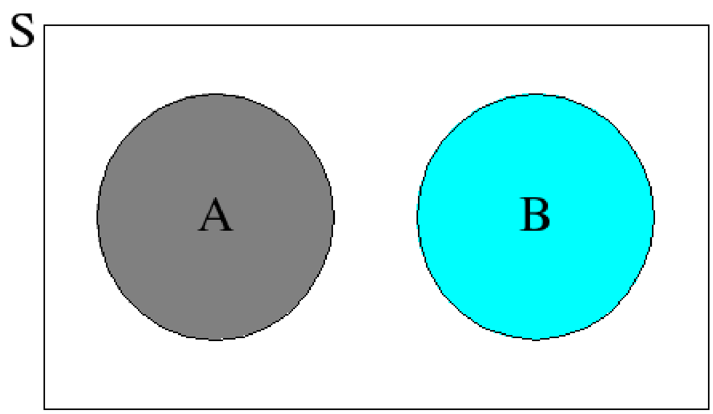
We see that there are no outcomes that are in both A and B. Therefore we can say that A and B are disjoint (mutually exclusive). Therefore $P(A \cap B) = 0$.

In general for two events $P(C \cup D) = P(C) + P(D) - P(C \cap D)$. We can see that this is so by the following Venn diagram:



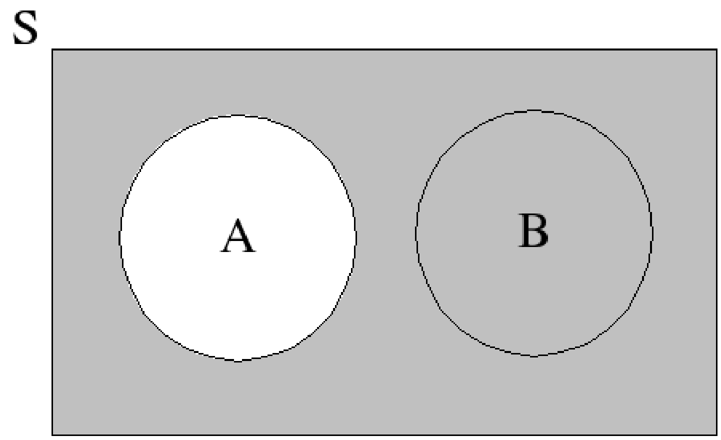
Notice that the overlap (intersection) would be counted twice so we must subtract it once.

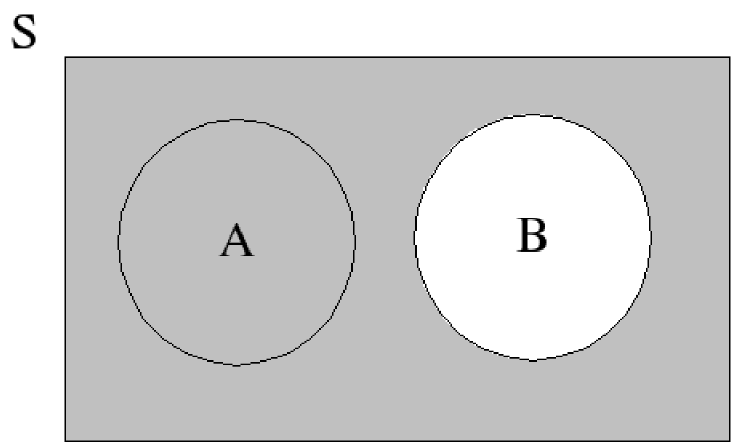
In our case there are no overlaps. We can represent it with the following Venn diagram.



Thus in our case $P(A \cup B) = P(A) + P(B) = \frac{1}{9} + \frac{1}{6} = \frac{5}{18}$.

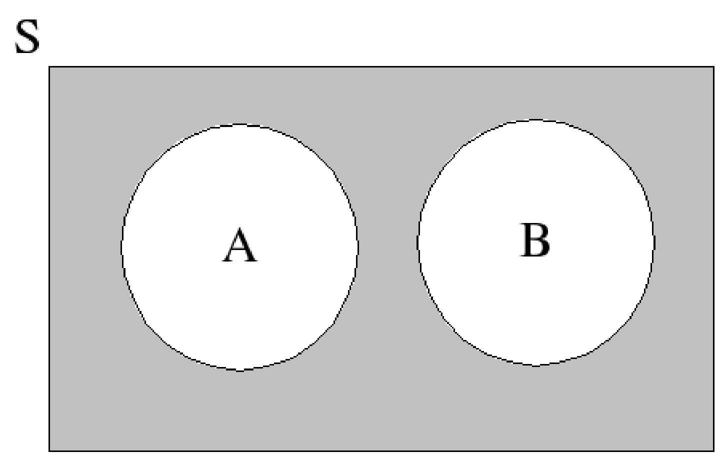
We examine complements lets first look at some Venn diagrams (the complements are given by the shaded areas). First A^c and then B^C:





Now remember that $P(A) + P(A^c) = 1$ therefore $P(A^c) = 1 - P(A) = 1 - \frac{1}{9} = \frac{8}{9}$. Similarly $P(B^C) = 1 - P(B) = 1 - \frac{1}{6} = \frac{5}{6}$.

Again we will use a Venn diagram to look at $P(A^c \cap B^c)$.



where we see that the shaded regions that are outside both A and B is $A^c \cap B^c$. Therefore $P(A^c \cap B^c) = 1 - P(A) - P(B) = 1 - 1/6 - 1/9 = 13/18$.

Finally, $P(A^c \cup B^c) = P(A^c) + P(B^c) - P(A^c \cap B^c) = \frac{8}{9} + \frac{5}{6} - \frac{13}{18} = 1$.

Question 2

A box has 6 balls. 4 are red and 2 are green. What is the probability of drawing first a red and then a green from the box (suppose that we are drawing at random and can't see inside the box).

1. Under replacement

2. Without replacement

Answer: Under replacement the outcome of the second drawing is independent from the outcome of the first drawing. Therefore, $P(RG) = P(R)P(G) = \frac{2}{3}\frac{1}{3} = \frac{2}{9}$.

Without replacement the outcome of the second drawing is dependent on the outcome from the first drawing. In particular, one the second drawing there will only be five balls left and if we draw a red on the first draw then there will be only three red balls remain for the second draw. Therefore $P(RG) = P(R)P(G|R\text{on first}) = \frac{2}{3}\frac{2}{5} = \frac{4}{15}$

Question 3

We have 4 boxes. Each contains 4 balls. In particular the contents of each of the boxes is as follows:

Box	Contents
1	2 White, 2 Red
2	1 White, 3 Red
3	3 White, 1 Red
4	1 White, 2 Red, 1 Green

Draw 1 ball from each box. What is the probability that at most 1 white ball is selected?

Answer: Let X be the number of white balls drawn from the four boxes.

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

Looking at the two components on the RHS of the above equation. Let R_i mean a red from box i. Then

$$\begin{aligned} P(X = 0) &= P(R_1R_2R_3R_4) + P(R_1R_2R_3G_4) \\ &= \frac{2}{4}\frac{3}{4}\frac{1}{4}\frac{2}{4} + \frac{2}{4}\frac{3}{4}\frac{1}{4}\frac{1}{4} \\ &= \frac{18}{256} \end{aligned}$$

and

$$
\begin{aligned}
P(X=0) &= P(W_1R_2R_3R_4) + P(W_1R_2R_3G_4) + P(R_1W_2R_3R_4) + P(R_1W_2R_3G_4) \\
&+ P(R_1R_2W_3R_4) + P(R_1R_2W_3G_4) + P(R_1R_2R_3W_4) \\
&= \frac{2}{4}\frac{3}{4}\frac{1}{4}\frac{2}{4} + \frac{2}{4}\frac{3}{4}\frac{1}{4}\frac{1}{4} + \frac{2}{4}\frac{1}{4}\frac{1}{4}\frac{2}{4} + \frac{2}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4} \\
&+ \frac{2}{4}\frac{3}{4}\frac{3}{4}\frac{2}{4} + \frac{2}{4}\frac{3}{4}\frac{3}{4}\frac{1}{4} + \frac{2}{4}\frac{3}{4}\frac{1}{4}\frac{1}{4} \\
&= \frac{84}{256}
\end{aligned}
$$

and so

$$P(X \leq 1) = P(X=0) + P(X=1) = \frac{18}{256} + \frac{84}{256} = \frac{102}{256}$$

Question 4

We now return to our coin tossing game of a previous class. The following table gives us the outcomes and probabilities for this game.

Outcome	-4	-2	0	2	4
Probability	1/16	1/4	6/16	1/4	1/16

Let X be a random variable representing the amount of our winnings from playing the game. What is the mean of this random variable? What is the variance? What is the standard deviation?

Answer: The formula for the mean (we can this the expected value) of a discrete random variable is

$$E(X) = \sum_i x_i p_i$$

in our case annotating our outcomes

Outcome	-4 (x_1)	-2 (x_2)	0 (x_3)	2 (x_4)	4 (x_5)
Probability	1/16 (p_1)	1/4 (p_2)	6/16 (p_3)	1/4 (p_4)	1/16 (p_5)

and therefore the expectation is given by

$$E(X) = -4(1/16) + -2(1/4) + 0(6/16) + 2(1/4) + 4(1/16) = 0$$

The formula for the variance of a discrete random variable is

$$
\begin{aligned}
Var(X) &= \sum_i (x_i - E(X))^2 p_i \\
&= (-4-0)^2(1/16) + (-2-0)^2(1/4) + 0^2(6/16) + (2-0)^2(1/4) + 4^2(1/16) \\
&= 4
\end{aligned}
$$

The standard deviation is the square root of the variance so

$$SD(X) = \sqrt{Var(X)} = \sqrt{4} = 2$$

Question 5

Suppose we play the same game as in Question 4 but with an unfair coin where P(H) = 0.25. What is E(X)? Var(X)? SD(X)?

Answer: First lets look the probability of each of the outcomes.

Outcome	-4	-2	0	2	4
Probability	$(3/4)^4$	$4(3/4)^3(1/4)$	$6(3/4)^2(1/4)^2$	$4(3/4)(1/4)^3$	$(1/4)^4$

thus the expectation is given by

$$\begin{aligned} E(X) &= -4(3/4)^4 + -2(4)(3/4)^3(1/4) + 0(6)(3/4)^2(1/4)^2 \\ &+ 2(4)(3/4)(1/4)^3 + 4(1/4)^4 \\ &= -2 \end{aligned}$$

and the variance is given by

$$\begin{aligned} Var(X) &= (-4 - -2)^2(3/4)^4 + (-2 - -2)^2(4)(3/4)^3(1/4) + (0 - -2)^2(6)(3/4)^2(1/4)^2 \\ &+ (2 - -2)^2(4)(3/4)(1/4)^3 + (4 - -2)^2(1/4)^4 \\ &= 3 \end{aligned}$$

and the standard deviation is

$$SD(X) = \sqrt{Var(X)} = \sqrt{3}$$