Stat 20 Fall 2003, Quiz 1 Answers

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Question 1. (*10 points*)

We have a box with 7 tickets. Each ticket is labeled with a distinct number from 1 to 7. We draw two tickets from the box. What is the probability of picking a prime on the first draw {2,3,5,7} and a divisor of 6 on the second draw {1,2,3,6}?

(a) With replacement?

Answer: With replacement we are free to pick any ticket on both the first and second draws. Let event $A =$ Pick one of $\{2, 3, 5, 7\}$ on 1st draw and let event $B =$ Pick one of $\{1, 2, 3, 6\}$ on 2nd draw. Thus the probability is

$$
\begin{aligned}
P(A \cap B) &= P(A)P(B) \\
&= \left(\frac{4}{7}\right)\left(\frac{4}{7}\right) \\
&= \frac{16}{49}
\end{aligned}
$$

(b) Without replacement?

Answer: When we sample with out replacement we can not pick the same ticket twice. In particular, if we pick a 2 or 3 on the 1st draw we cannot pick it on the 2nd draw. There are three cases which lead to the desired outcome.

- Pick one of {5,7} on first draw and then one of {1,2,3,6} on the second draw.
- Pick {2} on first draw and then one of {1,3,6} on the second draw.
- Pick {3} on first draw and then one of {1,2,6} on the second draw.

Each of these cases is a disjoint so we can add the probabilities together to get the probability of the outcome we are interested in.

$$
\begin{aligned}
P(\{5,7\} \text{ then } \{1,2,3,6\}) = P(\{5,7\})P(\{1,2,3,6\}|\{5,7\}) &= \left(\frac{2}{7}\right)\left(\frac{4}{6}\right) = \frac{8}{42} \\
P(\{2\} \text{ then } \{1,3,6\}) = P(\{2\})P(\{1,3,6\}|\{2\}) &= \left(\frac{1}{7}\right)\left(\frac{3}{6}\right) = \frac{3}{42} \\
P(\{3\} \text{ then } \{1,2,6\}) = P(\{3\})P(\{1,2,6\}|\{2\}) &= \left(\frac{1}{7}\right)\left(\frac{3}{6}\right) = \frac{3}{42}
\end{aligned}
$$

And so the probability of picking one of {2,3,5,7} on the first draw and then one of {1,2,3,6} on the second draw when we sample with out replacement is

$$\frac{8}{42} + \frac{3}{42} + \frac{3}{42} = \frac{14}{42} = \frac{1}{3}$$

Question 2. (*10 points*)

Suppose a gambler plays a game. The game cost \$1 to play. She receives \$5 (her initial bet plus \$4) if she wins, receives back \$0 if she loses. The probability of winning is 0.2. Each game is independent. Suppose the game is played 10 times. What is the gambler's expected winnings? What is the standard deviation of the gambler's winnings? Would you say the game is "fair", explain why or why not?

Answer using approach 1:

Let W_i be the net winning on the i'th game. Let T be the total winnings after 10 games.

$$E\left[W_i\right] = -1(0.8) + 4(.2) = 0$$

$$Var\left(W_i\right) = E\left[W_i^2\right] - \left(E\left[W_i\right]\right)^2$$
$$E\left[W_i^2\right] = (-1)^2(0.8) + (4)6(.2) = 4$$

and so

$$Var\left(W_i\right) = 4$$
$$SD\left(W_i\right) = \sqrt{Var\left(W_i\right)} = 2$$

Now let $T = W_1 + W_2 + \cdots + W_{10}$ and so

$$\begin{aligned} E\left[T\right] &= E\left[W_1 + W_2 + \cdots + W_{10}\right] \\ &= E\left[W_1\right] + E\left[W_2\right] + \cdots + E\left[W_{10}\right] \\ &= 10E\left[W_1\right] \\ &= 10[0] \\ &= 0 \end{aligned}$$

$$\begin{aligned} Var\left[T\right] &= Var\left[W_1 + W_2 + \cdots + W_{10}\right] \\ &= Var\left[W_1\right] + Var\left[W_2\right] + \cdots + Var\left[W_{10}\right] \\ &= 10Var\left[W_1\right] \\ &= 10[4] \\ &= 40 \end{aligned}$$

and so

$$SD\left(T\right) = \sqrt{40}$$

Answer using approach 2:

Let X be the number of times the gambler wins in 10 games. Then X is Binomial with $n = 10$ and $p = 0.2$. The Expected number of times the gambler wins is

$$E\left[X\right] = np = 10(0.2) = 2$$

The variance of the number of times the gambler wins is

$$E\left[X\right] = np(1-p) = 10(0.2)(0.8) = 1.6$$

Since we win \$4 dollars for each win and lose \$1 for each loss our total winnings in 10 games is

$$T = 4X - 1(10 - X) = 5X - 10$$

and so

$$E[T] = 5E[X] - 10 = 5(2) - 10 = 0$$

$$Var[T] = Var[5X - 10] = 5^2 Var(X) = 25(1.6) = 40$$

Is the game "fair"?

The game is fair because the expected winnings is \$ 0. That is the gambler does not lose anything, on average, if they play the game many times.

Question 3. (*5 points*)

Assume that X and Y are independent normal random variables with means 2 and 4 respectively and standard deviations 2 and 1 respectively. What is $P(X > Y)$?

Answer: Since X and Y are independent and X and Y are both normally distributed the following apply

1. $E[X - Y] = E[X] - E[Y] = 2 - 4 = -2$ (Note that this applies irrespective of independence or normality.)

2. $Var(X - Y) = Var(X) + Var(-Y) = Var(X) + (-1)^2 Var(Y) = 4 + 1 = 5$ (Note that this depends on independence).

3. $X - Y$ will be normally distributed with mean -2 and sd $\sqrt{5}$. (note this depends on both normality and independence)

So returning to our problem

$$\begin{aligned}
P(X > Y) &= P(X - Y > 0) \\
&= P\left(\frac{X - Y - -2}{\sqrt{5}} > \frac{0 - -2}{\sqrt{5}}\right) \\
&= P\left(Z > \frac{2}{\sqrt{5}}\right) \\
&= P(Z > 0.89) \\
&= 1 - P(Z < 0.89) \\
&= 1 - .8133 = 0.1867
\end{aligned}$$