Stat 215b (Spring 2004): A few words about ellipses

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This document should give you some idea's about how to draw the confidence ellipse for Lab 2.

The general ellipse equation

The general equation of an ellipse is given by

$$(\mathbf{x} - \mathbf{x_0})' \mathbf{Q} (\mathbf{x} - \mathbf{x_0}) \leq k$$

where $\mathbf{x_0}$ is a point in n dimensions representing the center of the ellipse, Q is a symmetric $n \times n$ matrix which determines the shape of the ellipse (orientation and how it is extended in various dimensions) and k is a scalar controlling the size.

Deriving the confidence ellipses

In the linear model context we look at the F-statistics to derive the equations for the ellipsoids. For notational purposes let $\beta = (\beta_0, \ldots, \beta_{p-1})$.

Consider linear tests of the form

$$H_0 : C\beta - \gamma = 0$$

where C is an $m \times p$ matrix of rank m. It can be shown that the above hypothesis will be rejected at level α if the statistic

$$F = \frac{\left(C\hat{\beta} - \gamma\right)' \left[C\left(X'X\right)^{-1} C'\right]^{-1} \left(C\hat{\beta} - \gamma\right)}{(y - X\hat{\beta})'(y - X\hat{\beta})} \frac{n-p}{m}$$

is such that $F > F^{(1-\alpha)}_{m,n-p}$.

Using the above it is easy to define confidence regions. For a confidence interval of level $1 - \alpha$ for the entire parameter vector β the confidence ellipsoid set $C = I_p$ and therefore the confidence region is given by

$$\frac{\left(\hat{\beta} - \beta\right)' (X'X) \left(\hat{\beta} - \beta\right)}{(y - X\hat{\beta})'(y - X\hat{\beta})} \frac{n-p}{p} \leq F^{(1-\alpha)}_{p,n-p}$$

As you might note this almost in the form of the ellipsoid equation above. Note that $s^2 = (y - X\hat{\beta})'(y - X\hat{\beta})/(n-p)$ and so we could write the ellipse as

$$\left(\hat{\beta} - \beta\right)' (X'X) \left(\hat{\beta} - \beta\right) \leq ps^2 F_{p,n-p}^{(1-\alpha)}$$

which is in the form above.

If we were instead interested in a confidence region for a subset of the parameters, say just the final two parameters then you would set $C = [0 I_2]$ where I_2 is the 2×2 identity matrix and 0 is the null matrix. Following the above derivation we get the corresponding confidence ellipse is

$$\left(\hat{\beta}_s - \beta_s\right)' D \left(\hat{\beta}_s - \beta_s\right) \leq 2s^2 F_{2,n-p}^{(1-\alpha)}$$

where D is the inverse of lower 2×2 sub matrix of $(X'X)^{-1}$ and $\beta_s = (\beta_{p-2}, \beta_{p-1})'$.

Similarly, for a confidence region for β_0 and $\beta_1 + \beta_2$ you might set

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \end{bmatrix}$$

and follow along in a similar manner to above.

Drawing a 2 dimensional ellipse

The general form of the ellipse equation is what is know as the *quadratic form.* In particular for the equation

$$(\mathbf{x} - \mathbf{x_0})' \mathbf{Q} (\mathbf{x} - \mathbf{x_0}) \leq k$$

in 2 dimensions we can use eigen analysis to draw the ellipse. Any introductory linear algebra text book should show how this is done. Some suggested R code is the following

```
V1 <- eigen(Q)$vec[,1]
V2 <- eigen(Q)$vec[,2]
e1 <- sqrt(1/eigen(Q)$val[1]*k)
e2 <- sqrt(1/eigen(Q)$val[2]*k)
theta <- (0:100)*2*pi/100
x <- e1*v1[1]*cos(theta) + e2*v2[1]*sin(theta) + x0[1]
y <- e1*v1[2]*cos(theta) + e2*v2[2]*sin(theta) + x0[2]
lines(x,y)
```